Algebraic Equivalence

Total time: 150 minutes but could be shortened to 75 minutes if needed. See page 2 for recommendations for a 75-minute module.

Situating the Module within the PTMT Teaching Algebra Materials: This module could be used alongside Chapter 4 Equivalence and Equality of the PTMT Algebra Materials. Specifically, Sections 2 and 4 which are focused on the use of CAS to explore equivalence, and Section 3 which uses virtual algebra tiles to explore geometric connections to algebraic equivalence. Sections 2 and 4 offer an explicit introduction to CAS tools that might be helpful to teachers who have not used CAS before. In addition, it offers other equivalence related topics for which CAS tools are helpful. Section 3 could be used alongside task 2.1 as it offers additional virtual manipulative examples that could be compared and contrasted.

Overview of the Module: This module focuses on the use of algebra tile virtual manipulatives and CAS to investigate algebraic equivalence in the context of completing the square. Teachers will compare the development of students' understanding of completing the square with the two math action technologies. For homework and in class the following day, they will view pairs of students engaging with each task as they practice their noticing skills with a focus on anticipating how students will engage with the two technologies and considering assessing and advancing questions to pose to students as a means of responding to interpretations of students' current understanding.

Module Goals:

- Consider how to use Polypad's Virtual Algebra Tiles in Desmos as a visual representation and investigation of completing the square.
- Consider how to use GeoGebra's CAS features as an algebraic representation and investigation of completing the square.
- Examine student practices and practice anticipating, attending and interpreting student thinking when working with these two technologies when completing the square.
- Compare and contrast two different types of technology and how the ways in which the models and expressions show equivalency between the given expression and the completed square.
- Compare and contrast the various ways in which students generalized the patterns they noticed in their models / expressions to create a rule for completing the square.

Table 1: Timeline of Tasks in the Module

Timeline of tasks in the Module	Day 1	May Vary	Optional Pre-Work with Algebra Tiles
		45 min	2.1 Completing the Square with Algebra Tiles
		20 min	2.2 Completing the Square with CAS
		10 min	Compare and Contrast 2.1 and 2.2
		Homework	2.3 Noticing Student Thinking: Completing the Square with Virtual Algebra Tiles
	Day 2	15 min	2.3 Discussion
		20 min	2.4 Posing Purposeful Questions: Completing the Square with Virtual Algebra Tiles
		30 min	2.5 Noticing Student Thinking: Completing the Square with CAS
		10 min	Revisit Compare and Contrast 2.1 and 2.2
		Homework	Read the following two articles: Lischka, A. E., & Stephens, D. C. (2020). The area model: Building mathematical connections. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(3), 186–195. Zbiek, R. M., Heid, M. K. & Hirsch, C. (2009). Using computer algebra systems to develop big ideas in mathematics. *Mathematics Teacher*, *(102)*7, 540–544.

Recommendations for 75-minute module: This module is designed for 150 minutes but could easily be shortened to 75 minutes if needed. One approach is to choose one of the two technology tasks for teacher investigation. It may be that your teachers could benefit from working with Polypad's Virtual Algebra Tiles in Desmos, or perhaps they would benefit more from a CAS exploration. Another approach is to have teachers engage with 2.1 outside of class but this is only recommended if teachers have significant previous experience with algebra tiles.

MTE preparation for Module 2:

- Read NCTM's *Taking Action Grades 6-8* or *Taking Action 9-12*, Chapter 5: Pose Purposeful Questions.
- Engage with both completing the squares tasks: 2.1 Completing the Square with Algebra Tiles and 2.2 Completing the Square with CAS.
- Read facilitation notes and commentaries, engage with assignments (including watching associated videos), and look over sample responses.
- For a deeper dive into the mathematical ideas, see pages 52–53 from Cooney, T. J., Beckmann, S., Lloyd, G. M., & Wilson, P. S. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12.* National Council of Teachers of Mathematics.
- For a deeper dive into computer algebra systems (CAS) see
    - Mahoney, J. F. (2002). Computer algebra systems in our schools: Axioms and some examples. *Mathematics Teacher (95)*8, 598–605.
    - Heid, M. K., Blume, G. W., Hollebrands, K. F., & Piez, C. (2002). Connecting research to teaching: Computer algebra systems in mathematics instruction: Implications from research. *Mathematics Teacher, (95)*8, 586–591.

Suggested Readings for Teachers:

- Lischka, A. E., & Stephens, D. C. (2020). The area model: Building mathematical connections. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(3), 186–195.
- Zbiek, R. M., Heid, M. K. & Hirsch, C. (2009). Using computer algebra systems to develop big ideas in mathematics. *Mathematics Teacher*, *(102)*7, 540–544.
- Richardson, J., & Bachman, R. M. (2017). A new take on an old square. *Mathematics Teacher, 110*(9), 668–673.

Detailed Agenda for Module 2

Table 2: Agenda

	Description of Module Tasks	**Facilitation Notes**
Day 1	**Optional Pre-Work with Algebra Tiles** (Timing will vary)	Depending on the teachers' familiarity with algebra tiles as a manipulative (physical or virtual), we recommend spending class time using algebra tiles to model multiplication, division, and factoring prior to tackling completing the square.
	2.1 Completing the Square with Algebra Tiles (45 minutes) Teacher devices Desmos Task Teachers complete the Desmos task that has them engage Polypad's virtual algebra tiles to visually complete the square.	Create a class code for the Desmos task and provide the link to your students. See the full Instructor Materials which includes suggestions for pacing the Desmos task, sample responses, and a link for a video podcast on completing the square with algebra tiles.

Day 1 (cont.)	**2.2 Completing the Square with CAS** (20 minutes) Teacher devices Worksheet Teachers complete a worksheet that uses GeoGebra's CAS tools to investigate completing the square from an algebraic perspective.	Provide a copy of the electronic worksheet to the teachers and work through the examples at the top of the page as a whole class to ensure the teachers understand how to use GeoGebra's factor and expand tools. Let the teachers know they will use these tools to look for patterns in factored quadratic expressions. We recommend having a whole class discussion after teachers complete the first section to share what they notice and wonder. Then display the area model of a perfect square and discuss how their ideas connect to the model. After teachers complete the task, facilitate a whole class discussion around their recorded generalizations. See the full Instructor Materials which include the facilitation notes and sample responses.
	Compare and Contrast 2.1 & 2.2 (10 minutes)	Provide teachers with time to discuss similarities and differences between 2.1 and 2.2. Specifically, have teachers discuss the extent to which each task is connected to the area model of completing the square and how the tasks relate to various algebraic expressions. As time allows, have teachers briefly share with the whole class. Let the teachers know they will return to this discussion after examining student work on both tasks for homework and in the following class period. See the full Instructor Materials which include the facilitation notes and sample responses.

Day 1 / Homework	**2.3 Noticing Student Thinking: Completing the Square with Virtual Algebra Tiles** Teacher devices Worksheet Teachers notice one pair of students' mathematical thinking as they work on page 10 of the Desmos Algebra Tiles Task.	Provide a copy of the electronic worksheet to the teachers. We recommend pointing teachers to the NITE framework and reminding teachers that when noticing student thinking with technology, they should pay careful attention to what students' say, do with the technology, and what they record. Teachers will read NCTM's *Taking Action 9-12 Chapter 5: Pose Purposeful Questions* See the full Instructor Materials which include the facilitation notes and sample responses.
Day 2	**2.3 Discussion** (15 minutes)	Provide teachers with time to discuss their responses in small groups. We recommend first focusing on Q2, Q4, & Q8 and then having a whole class discussion focused on Q9 & Q10. See the Instructor Materials for details.
	2.4 Posing Purposeful Questions: Completing the Square with Virtual Algebra Tiles (20 minutes) Teacher devices Worksheet Teachers practice posing purposeful assessing and advancing questions to pairs of students who have used Polypad differently to model the same expression.	Provide a copy of the electronic worksheet to the teachers. We suggest teachers work in small groups to complete Q1–Q4. Prior to answering Q5, we recommend stopping for a whole-class discussion making sure to highlight connections between the models and expressions each pair of students produced as well as the various ways in which algebra tiles were used to model the expression. Have teachers share how their proposed assessing and advancing questions differed based on the students' various models. See the full Instructor Materials which include the facilitation notes and sample responses.

	2.5 Noticing Student Thinking: Completing the Square with CAS (30 minutes) Teacher devices Worksheet Teachers practice noticing, including anticipating, attending, interpreting and finally posing purposeful assessing and advancing questions, for a pair of students who have used CAS to develop an understanding of completing the square.	Provide teachers with an electronic copy of the worksheet and assign pairs to complete the task. A whole class discussion is recommended to collect details of teachers' attend and interpret statements (Q2 and Q3), with a focus on coordinating the spoken and written aspects with the aspects related to the students' technology engagement. Once those details have been collected and coordinated, open up the discussion to their responses to Q4. Invite them to add or refine their questions based on the discussion of Q2 and Q3. See the full Instructor Materials which include the facilitation notes and sample responses.

Day 2 (cont.)	**Compare and Contrast 2.1 and 2.2** (10 minutes)	Provide teachers with time to once again discuss similarities and differences between 2.1 and 2.2. Specifically, have teachers discuss how the different technologies helped build students' understanding of completing the square in different ways. The focus of the discussion should be on student thinking and benefits/drawbacks of engaging students in one or both tasks. See the full Instructor Materials which include the facilitation notes and sample responses.
Day 2 / Homework	**Read:** 🔗 Lischka & Stephens (2020). The area model: Building mathematical connections 🔗 Zbiek & Heid (2009) Using Computer Algebra Systems to Develop Big Ideas in Mathematics	Teachers will read the following two articles: Lischka, A. E., & Stephens, D. C. (2020). The area model: Building mathematical connections. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(3), 186–195. Zbiek, R. M., Heid, M. K., & Hirsch, C. (2009). Contemporary curriculum issues: Using computer algebra systems to develop big ideas in mathematics. *Mathematics Teacher*, *(102)*7, 540–544.